Tutorial 7 : Selected problems of Assignment 7

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Recall the Perturbation of Identity:

Thm Let $(X, \|\cdot\|)$ be a Banach space (i.e. complete normed space), and $\Phi: X \to X$ be a continuous map w/ $\Phi(x_0) = y_0$

If $\exists\, r>0$ s.t. $\Phi\big|_{\overline{B_r(x_0)}} : \overline{B_r(x_0)} \to X$ has the form

$\Phi = I + \Psi$, where $I = Id_X : X \to X$ is the identity map

and $\Psi : \overline{B_r(x_0)} \to X$ is a contraction with constant $\gamma \in (0,1)$

then $\Phi : \overline{B_r(x_0)} \to \overline{B_R(y_0)}$ is uniquely solvable with $R = (1-\gamma)r$

i.e. $\forall\, y \in \overline{B_R(y_0)}$, $\exists!\, x \in \overline{B_r(x_0)}$ s.t. $\Phi(x) = y$

Q1) (HW7, Q4)

Show that $\sin^2 \pi x + 2x^2 = 2.0012$ is solvable near $x = 1$.

Sol: Step 1: manipulate the equation to the form $x + \Psi(x) = y$

$(X, \|\cdot\|) = (\mathbb{R}, |\cdot|)$; define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin^2 \pi x + 2x^2$

Define $\Phi: \mathbb{R} \to \mathbb{R}$ by $\Phi(x) = \frac{1}{4} f(x+1)$, then $\Phi(0) = \frac{1}{2} =: y_0$

By definition, $\Phi(x) = \frac{1}{4}\left(\sin^2 \pi(x+1) + 2(x+1)^2\right)$

$= \frac{1}{4}\sin^2 \pi x + \frac{x^2}{2} + x + \frac{1}{2} = x + \Psi(x)$, where $\Psi(x) = \frac{1}{4}\sin^2 \pi x + \frac{x^2}{2} + \frac{1}{2}$

$\therefore$ Equivalently, we show the solvability of $\Phi(x) = \frac{1}{2} + 0.0003$ near $x_0 = 0$.

Step 2: Determining the contractibility of Ψ

$$\Psi(x) = \frac{1}{4}\sin^2 \pi x + \frac{x^2}{2} + \frac{1}{2}\ ; \quad \Psi'(x) = \frac{\pi}{4}\sin 2\pi x + x$$

$\therefore\ \forall\, r > 0,\ \forall\, z \in \overline{B_r(0)},\ |\Psi'(z)| \leq \left(\frac{\pi}{4}|2\pi z| + |z|\right) \leq \left(\frac{\pi^2}{2} + 1\right) r$

$\therefore\ \forall\, x, x' \in \overline{B_r(0)},\quad |\Psi x - \Psi x'| = |\Psi'(z)|\,|x - x'|,\ \exists\, z \in \overline{B_r(0)}$

$$\leq \left(\left(\frac{\pi^2}{2} + 1\right) r\right) |x - x'|$$

$\therefore\ \forall\, 0 < r < \frac{1}{\frac{\pi^2}{2} + 1} = \frac{2}{\pi^2 + 2}$, $\Psi: \overline{B_r(0)} \to \mathbb{R}$ is a contraction.

Step 3: Apply the Theorem by choosing a suitable $r>0$

$$\gamma = \left(\frac{\pi^2}{2}+1\right) r \ ; \quad R := (1-\gamma) r = \left(1-\left(\frac{\pi^2}{2}+1\right) r\right)\cdot r$$

Want to fix $r>0$ s.t. $\frac{1}{2}+0.0003 \in \overline{B_R(\frac{1}{2})}$, i.e. $R > 0.0003$

By trial and error, $r = \frac{1}{4\pi^2}$ will work:

$$0 < r < \frac{2}{\pi^2+2} \quad \text{and} \quad R \approx 0.0215 > 0.0003$$

$\therefore$ By the Thm, $\Phi(x) = \frac{1}{2}+0.0003$ is solvable near $x_0 = 0$

which implies $f(x) = 2.0012$ is solvable near $x=1$. $-\square$

Q2) (HW7, Q7) Show that the integral equation over $C[-1,1]$

$$y(x) = \alpha e^x - \int_0^1 \frac{\sin x}{3-t} y^3(t)\,dt$$

is solvable near $y(x) \equiv 0$ for sufficiently small α.

Sol: Let $(X, \|\cdot\|) = (C[-1,1], \|\cdot\|_\infty)$, following the steps as in Q1:

Step 1: $y(x) + \int_0^1 \frac{\sin x}{3-t} y^3(t)\,dt = \alpha e^x$

$\therefore$ Define $\Phi: X \to X$ by $\Phi(y(x)) = y(x) + \int_0^1 \frac{\sin x}{3-t} y^3(t)\,dt$

then $\Phi(0) = 0$; also $\Phi = I + \Psi$, where $\Psi(y(x)) = \int_0^1 \frac{\sin x}{3-t} y^3(t)\,dt$

Step 2: $\forall r > 0$, $\forall y_1, y_2 \in \overline{B_r(0)}$, $\forall x \in [-1,1]$, $|\Psi(y_1(x)) - \Psi(y_2(x))|$

$= \left| \int_0^1 \frac{\sin x}{3-t} (y_1^3(t) - y_2^3(t))\,dt \right| \leq \int_0^1 \frac{1}{2} \|y_1 - y_2\|_\infty \|y_1^2 + y_1 y_2 + y_2^2\|_\infty dt$

$\leq \frac{3r^2}{2} \|y_1 - y_2\|_\infty$ $\therefore$ When $0 < r < \sqrt{\frac{2}{3}}$, then Ψ is a contraction.

Step 3: $\gamma = \frac{3r^2}{2} < 1$; $R = (1-\gamma) r = (1 - \frac{3r^2}{2}) r$

Since $\|\alpha e^x - 0\|_\infty = |\alpha| e$, by the Theorem, when $|\alpha| < \frac{R}{e} = \frac{1}{e}(1 - \frac{3r^2}{2}) r$

then the integral equation is solvable on $\overline{B_r(0)}$, where $0 < r < \sqrt{\frac{2}{3}}$.

Q3) (HW7, Q8) Let $A = (a_{ij})$ be real $n\times n$ matrix,

(a) Show that if $\sum_{i,j} a_{ij}^2 < 1$, then $I+A$ is invertible.

(b) Give an example which $\sum_{i,j} a_{ij}^2 = 1$ and $I+A$ is singular.

Sol: (a) $(X, \|\cdot\|) = (\mathbb{R}^n, \|\cdot\|_2)$; let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be defined as

$\Phi(x) = x + Ax$, then $\Phi(0) = 0$: $\Phi = I + \Psi$, where $\Psi(x) = Ax$

Showing Ψ is a contraction: $\forall\, x, x' \in \mathbb{R}^n$,

$$\|\Psi(x) - \Psi(x')\|_2 = \|A(x-x')\|_2 \leq \|A\|\,\|x-x'\|_2 \leq \left(\sum_{i,j} a_{ij}^2\right)\|x-x'\|_2$$

$\therefore$ Choose $\gamma = \sum_{i,j} a_{ij}^2 < 1$, Ψ is a contraction on $\mathbb{R}^n$.

$\therefore$ By the theorem, $\forall r > 0$, $\Phi(x) = 0$ is uniquely solvable on $\overline{B_r(0)}$

Hence Φ is uniquely solvable on $\mathbb{R}^n$.

As $\Phi(0) = 0$, $\forall\, x \in \mathbb{R}^n$ s.t. $\Phi(x) = 0$, $x = 0$, i.e. $\operatorname{Ker}\Phi = \{0\}$

Therefore, $I+A$ is invertible.

(b) Take $n = 1$, $A = (-1)$, then $I+A = (0)$ is singular.